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Exam Three MTH–221 , Summer 2022

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Score = $\frac{46}{46}$ Excellent

QUESTION 1. (16 points)

(i) Let A be a 3×3 matrix such that $C_A(\alpha) = (\alpha - 2)(\alpha - 3)^2$, Given $E_3 = span\{(2, 2, 2), (-2, 2, 2)\}$, and $E_2 = span\{(-2, -2, 2)\}$. Then, we know there is an invertible matrix Q such that $Q^{-1}AQ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

One of the following is a possibility for Q.

(a) $\begin{bmatrix} 2 & 2 & 2 \\ -2 & 2 & 2 \\ -2 & -2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -2 & -2 \\ 2 & -2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -2 & -2 \\ 2 & 2 & -2 \\ 2 & 2 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 & 2 \\ -2 & -2 & 2 \\ -2 & 2 & 2 \end{bmatrix}$

(ii) Let $D = \left\{ \begin{bmatrix} a+2b & a+2b \\ -a-2b & b \end{bmatrix} \mid a, b \in R \right\}$. Then D is a subspace of $R^{2\times 2}$. A basis for D is

(a) $\left\{ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -2 & 0 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix} \right\}$

(iii) Let $T : R^3 \to P_3$ be a linear transformation such that $T(a, b, c) = (a-b+2c)x^2+(2a-2b+4c)x+(-a+b-2c)$. Then a basis for Range(T) is

(a) $\{(1, 0, 0)\}$ (b) $\{x^2\}$ (c) $\{x^2, x, 1\}$ (d) $\{x^2 + 2x - 1\}$

(iv) Let $T : R^{2\times 2} \to R_3$ be a linear transformation such that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a - b - c,\ 0,\ d))$. Then a basis for Z(T) is

(a) $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

(v) Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$. It is clear that 2, 4 are the eigenvalues of A. Then $E_2 =$

(a) span$\{(0, -4)\}$ (b) $\{(0, 0)\}$ (c) span$\{(2, 0)\}$ (d) span$\{(0, 1)\}$

(vi) Let $A = \begin{bmatrix} 4 & 4 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix}$. Then

(a) A^{-1} exists ✗ (b) $Rank(A) = 3$ ✗ (c) A is diagnolizable (d) the rows of A are independent ✗

(vii) One of the following is a subspace of P_4

(a) $\{(a + 2)x^3 + ax + a \mid a \in R\}$ (b) $\{x^2 + ax + b \mid a, b \in R\}$ (c) $\{ax^2 + (a + b)x + 6b \mid a, b \in R\}$
(d) $\{ax^4 + 3bx^2 + 2a + 2b \mid a, b \in R\}$

(viii) Given $T : R^2 \to R^2$ such that $T(a,b) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$ and $L : R^2 \to R^2$ such that $L(a,b) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$.
Then $(T \circ L)^{-1}$ exists. The standard matrix presentation of $(T \circ L))^{-1}$ is

(a) $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

QUESTION 2. Let $T : R^{2\times 2} \to P_3$ be an R-homomorphism (i.e., Linear Transformation) such that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-c-d)x^2 + (b-4d)x + (-b+4d)$

(i) (5 points) Find all matrices in $R^{2\times 2}$ such that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 4x^2 + 7x - 7$

$L : \mathbb{R}^4 \to \mathbb{R}^3$ $\quad L(a,b,c,d) = (a-c-d, b-4d, -b+4d)$

$$M_L = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 0 & 4 \end{bmatrix} \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 4 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & -1 & 0 & 4 & -7 \end{array}\right] \xrightarrow{R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 4 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Read:

$a - c - d = 4 \to a = 4 + c + d$

$b - 4d = 7 \to b = 7 + 4d$

$c, d \in \mathbb{R}$

Solution set in L: $\{(4+c+d, 7+4d, c, d) \mid c, d \in \mathbb{R}\}$

Solution set in T: $\left\{\begin{bmatrix} 4+c+d & 7+4d \\ c & d \end{bmatrix} \,\middle|\, c, d \in \mathbb{R}\right\}$

(ii) (5 points) Find a basis for $Z(T)$.

$Z(T) = \left\{\begin{bmatrix} c+d & 4d \\ c & d \end{bmatrix} \,\middle|\, c, d \in \mathbb{R}\right\}$

$Z(T) = \left\{c\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \,\middle|\, c, d \in \mathbb{R}\right\}$

$Z(T) = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\right\}$

check for independance:

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 4 & -1 & 1 \end{bmatrix}$

$\dim(Z(T))$ = num of free variables = 2

basis for $Z(T) = \left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\right\}$

QUESTION 3. (10 points) Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

If A is diagnolizable, then find a diagonal matrix D and an invertible matrix Q such that $Q^{-1}AQ = D$ (Do not find Q^{-1}).

$$C_A(\alpha) = \begin{vmatrix} \alpha-2 & 0 & 0 & 0 \\ -1 & \alpha-3 & 0 & 0 \\ -1 & 0 & \alpha-3 & 0 \\ -1 & 0 & 0 & \alpha-3 \end{vmatrix} = (\alpha-2)(\alpha-3)^3$$

$\alpha = 2$ (repeated once)

$\alpha = 3$ (repeated 3 times

$$E_2 = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow[-R_2+R_4 \to R_4]{-R_2+R_3 \to R_3} \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[-R_3+R_4 \to R_4]{R_3+R_2 \to R_2} \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_4+R_2 \to R_2]{R_4+R_3 \to R_3} \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Read:
$a = -d$ $\quad d \in \mathbb{R}$
$b = d$
$c = d$

sol $E_2 = \{(-d, d, d, d) \mid d \in \mathbb{R}\}$

$E_2 = \text{Span}\{(-1,1,1,1)\}$

$$E_3 = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\substack{R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3 \\ R_1+R_4 \to R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Read:
$a = 0$
$b, c, d \in \mathbb{R}$

$E_3 = \{(0, b, c, d) \mid b, c, d \in$

~~E_3 = Span~~

$E_3 = \{b(0,1,0,0) + c(0,0,1,0) + d(0,0,0,1) \mid b,c,d \in \mathbb{R}\} = \text{Span}\{(0,1,0,0), (0,0,1,0), (0,0,0,$

A is diagnolizable since $\dim(E_2) = 1 = \#$ of times 2 is repeated and $\dim(E_3) = 3 = \#$ of times 3 is repeated

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \longrightarrow Q = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

QUESTION 4. (i) (5 points) Convince me that $D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 0 \ \ and \ \ b - c = 0 \right\}$ is a subspace of $R^{2\times 2}$. Then find a basis for D.

$a = -d$ and $b = c$

$D = \left\{ \begin{bmatrix} -d & c \\ c & d \end{bmatrix} \middle| \ c, d \in \mathbb{R} \right\}$

$D = \left\{ c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \middle| \ c, d \in \mathbb{R} \right\}$

$D = \text{Span}\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

D is a subspace because it can be written as span.

Check for independance:

$\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

The points are independant, so:

Basis for $D = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(ii) (5 points) Let $D = \{f(x) \in P_4 \mid f(1) = f(-1) = 0\}$. Convince me that D is a subspace of P_4. Find a basis for D.

$-a_3 + a_2 - a_1 + a_0 = 0$

From the condition:

$a_3 + a_2 + a_1 + a_0 = -a_3 + a_2 - a_1 + a_0$

$\cancel{a_3 = -a_3 \to so\ a_3 = 0}$

$\cancel{a_1 = -a_1 \to so\ a_1 = 0}$

$2a_3 + 2a_1 = 0$

$a_3 = -a_1$

$f(1) = 0$

$-a_1 + a_2 + a_1 + a_0 = 0$

$\cancel{a_2 + a_0 = 0}$

$\cancel{a_2 = -a_0}$

$a_2 + a_0 = 0$

$a_2 = -a_0$

$\cancel{\{0x^3 - a_0x^2 + 0x + a_0 \mid a_0 \in \mathbb{R}\}}$

$\cancel{\{-a_0x^2 + a_0 \mid a_0 \in \mathbb{R}\}}$

~~$D = \text{Span}\{-x^2+1\}$ so D is a subspace~~

~~basis for $D = \{-x^2+1\}$~~

$D = \{-a_1x^3 - a_0x^2 + a_1x + a_0 \mid a_1, a_0 \in \mathbb{R}\}$

$D = \text{Span}\{-x^3 + x, -x^2 + 1\}$ so it is a subspace

Basis for $D = \{-x^3 + x, -x^2 + 1\}$